NPV Formulas

A Fast Method for Calculating and Optimising NPV in Screening Studies

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Summary

When quick estimates of NPV are needed for oil-field development options, it is usual to calculate the economics in a spreadsheet model, using a production profile generated by means of a set of simple assumptions (e.g. typical initial well rates, STOIIP, recovery factor etc). Because

- in most countries, oil-field economics can be reduced to linear factors (for tax effects) and discounting;
- discounting and exponential decline can be combined into a single exponential function;

it is possible to take a set of assumptions, similar to those used in a typical spreadsheet model, and derive the following equations (N.B. commercial use is subject to patent protection)

$$\text{Number of wells that maximises NPV} = \frac{R.d}{q}.\left(\sqrt{L - \frac{\alpha.E}{q}} - \sqrt{\frac{C.d + E - \alpha.E}{q}}\right) \Bigg/ \sqrt{\frac{C.d + E - \alpha.E}{q}}$$

$$\text{NPV with this number of wells} = R.\left(\sqrt{L - \frac{\alpha.E}{q}} - \sqrt{\frac{C.d + E - \alpha.E}{q}}\right)^2 - D$$

where $\alpha = (1+d)^{-Tab}$ and T_{ab} (abandonment time)= $\frac{R}{q.N}.\ln\left(\frac{L.q}{E}\right)$

and

q – initial oil production per well per year, averaged over all wells, including injectors

N – total number of wells, including injectors

R – technical reserves i.e. the amount of oil that could be recovered if the field were run for a very long time

L – net revenue per tonne of oil (i.e. after all taxes and royalties, including profit tax)

d – discount rate

C – net capital cost per well

D – net capital costs not related to numbers of wells, e.g. roads and pipelines

E – net opex per well

Note:- In this context, "net" means expressed in terms of effect on present value, after all taxes and royalties

Uses of the equations

These equations offer a number of advantages when used in addition to existing methods:-

- quick and easy to use, especially for screening prospects and initial concepts, in the early stages of a project
- easily auditable (in contrast to spreadsheets, which are often very difficult to fathom)
- can provide a totally independent check on project economics
- can help save time in determining the correct well density
- can help to provide an over-view of the common task of almost all petroleum engineers – optimising the economic return.

The equations can also be extended to give the optimal number of wells when the objective of the oil company is to maximise the overall rate of return.

In a trial, when applied to a Russian field, the equations quickly yielded very similar results to one month of conventional work.

An example

Consider a field with the following technical and economical parameters:-

Technically recoverable reserves = 100 million barrels

Net oil price = $8 per barrel

Discount rate = 8%

Net fixed capex costs = $20,000,000

Two development options are considered:-

a) Vertical wells
   Net capex per well = $2,000,000
   Net opex per well per year = $300,000
   Average initial well rates = 500 barrels per day = 150,000 barrels per year
b) Horizontal wells
   Net capex per well = $5,000,000
   Net opex per well per year = $500,000
   Average initial well rates = 1,500 barrels per day = 450,000 barrels per year

The formulas quickly give

a) Vertical wells
   Optimal number of wells = 35
   NPV with this number of wells = $122,000,000
   (Time to abandonment = 26 years)
b) Horizontal wells
   Optimal number of wells = 19
   NPV with this number of wells = $209,000,000
   (Time to abandonment = 23 years)

One can see that these results match the way that horizontal developments typically need fewer wells than vertical ones do.

Creating an analytic description of a typical economic model

In order to derive algebraic expressions for NPV and for the economically optimal number of wells, it is necessary to have an algebraic model of all the economic factors, including taxes and royalties. It is suggested that the following is an appropriate model for most tax regimes:-

$NPV = \Sigma\ (L.Qi + \alpha.OPEXi + \beta.CAPEXi) / (1+d)^{i-1}$

where the summation is over i=1 to i=n, the year of the end of the field-life

OPEXi = OPEX in year i

CAPEXi = CAPEX in year i, not including VAT

Qi = oil production in year i

L = a linear factor, the "net oil price" – it represents the benefit in present value terms of one extra unit (ton, cubic metre or barrel) of oil sold

α = a linear factor, representing the effects on NPV of one extra real-terms dollar (or whatever unit of currency) of expenditure on OPEX

β = a linear factor, representing the effects on NPV of one extra real-terms dollar of expenditure on CAPEX

The terms α.OPEXi and β.CAPEXi can be considered to be the "net OPEX" and the "net CAPEX".

It is reasonable to model the NPV as a linear function for two reasons

i) although, as can seen from economics spreadsheet models, the interplay of costs, revenues and taxation is very complicated, usually all the individual steps in the spreadsheets are linear; hence combining them will give a linear function

ii) if a taxation system was not linear in its effects, there would be benefits in artificially splitting or combining projects so as to take advantage of the non-linearities; such a possibility would quickly become well known. Hence, from initial design or from progressive correction of anomalies, most tax systems have evolved to be linear.

Other economic assumptions

The other major assumptions about the economics are that

- it is desired to achieve the highest possible NPV (as opposed to the highest rate of return, for example);
- capex expenditure for the field consists of a fixed element (e.g. cost of platform, road or pipeline) and a cost per well;
- all capex expenditure occurs in Year 1;
- opex is proportional to the number of wells.

In analytic models, it is usually necessary to define in advance what quantity is to be minimised or maximised. (In contrast, in spreadsheet models, it is possible to run through all the various scenarios and then choose the one which gives the overall best results. For example "Putting in 15 wells rather than 10 increases the NPV by only 5%, while it increases the capex by 40%. On balance, we think that the increase in NPV is not worth it, given the overall uncertainties, so we will go for the 10 well option.") In this article, NPV will be maximised, firstly because it is the simplest, and, secondly, because it is easy to modify the NPV-maximising results to cope with the situation where there is a shortage of capital, and one is only interested in returns of more than 15%, for example. The modification required is described later in the article.

The first assumption about capex are that net capex can be expressed as

Net capex = C.N + D

where C is net capex per well and D is a fixed element. The second assumption is that all the capex expenditure can be considered to take place instanteously at the beginning of the project. Clearly, this is not always realistic, but, again,

- it is simpler;
- it is unlikely that the time required to carry out the capital expenditure will affect questions such as "What is the optimal number of wells?"
- it is easy to modify the model to deal with capex spread over several years (again, the modification is described later in the article).

With regards to opex, it should be noted that, whenever the gross liquid rate is constant, the simple model gives the same results as a more complicated model, in which opex contains one element proportional to the number of wells and one element proportional to the gross liquid production.

Reserves, well rates and decline rates

There are three important petroleum engineering assumptions

- technically recoverable reserves are independent of the number of wells
- initial well rates are independent of the number of wells
- the field follows exponential decline from the start of production.

Unless there are geological features like isolated fault blocks, the number of wells does not much affect the technical recovery factor. In theory, for a field producing under depletion, a single well could, over a very long time, drain the field as efficiently as twenty wells. One reason for this is that, whatever the well spacing, the area of the field that is drilled up (i.e. directly penetrated by a well-bore) is very small in relation to the total area. With vertical wells, a well spacing of 500m and a well-bore diameter of 6 inches gives a ratio (total well bore area): (area of field) of 1×10^{-7}. With ten times as many wells, this would still only give a ratio of 1×10^{-6}, so similar recovery factors should apply.

It can be seen from the formula for the steady-state productivity index (PI) of a vertical well

$PI = (constant \times K_o.h) / [(\ln(r_e/r_w)+S) \times B_o.\mu_o]$

and the slightly more complicated formula for horizontal wells (Refs 1 and 2), that term depending on well separation, r_e (the effective drainage radius) only occurs as a logarithmic terms, hence the effect of changes of it are usually small. So it is reasonable to assume that PI is not much affected by changes in well spacing.

Even if PIs are constant, it is does not necessarily follow that well rates are constant, because one well may reduce the average pressure seen by the other wells. However, if one scales a whole development, including the number of injectors, then the average pressures should remain the same, giving constant well rates.

If fraccing is used, it may be incorrect to assume that PIs are independent of well numbers. Fraccing can be considered to give a large negative skin, so it can be the case that, even if the proportional change in $\ln(r_e/r_w)$ is small, the change in $[(\ln(r_e/r_w)+Skin]$ may be significant.

The assumption that *initial* well rates are independent of well numbers also holds for gas fields being produced under depletion drive. Here, increasing the number of wells does give pressure interference. However, providing all the wells start production at the same time, this effect is captured by the decline rate.

The other major petroleum engineering assumption is that the field follows exponential decline from the very beginning. There may be two objections to this asssumption. Firstly, many fields show a plateau production period. Secondly, once decline starts, it may be exponential, but changes to hyperbolic decline in the later stages of field life.

In answer to the first objection, it can be argued that many of the fields showing a production plateaus did so because of staged development. If this is the case, then the effect can be captured by the modification to the model described later. However, if the plateau period is a genuine subsurface phenomenon (maybe a period of stable, field-wide dry oil production, before water break-through), then the model may require further development.

The second objection should be less of a concern. A change to hyperbolic decline usually occurs sufficiently late in field life that, with discounting, it has little effect on the economics of the field, and on the question of what is the economically optimal number of wells. All that is required for the model to work correctly is to use a modification to the reserves – namely the total production at time=infinity if expontial decline had continued for the whole life of the field.

Derivation of the formulas

The derivation of the formulas was done in four steps

1) By simple integration, derive a formula for NPV as a function of (amongst other things) well numbers.
2) Calculate the optimal number of wells by taking partial derivatives with respect to number of wells and abandonment time.
3) Show that the formula for the optimal number of wells can be approximated by a simpler expression.

4) Calculate the NPV for the approximation of the optimal number of wells.

A full mathematical proof of these results is given in the appendix.

Understanding the optimal number of wells

One of the big advantages of formulas, rather than spreadsheets, is that they give more of an insight of how all the factors interact. Consider the formula for expressing NPV as a function of the number of wells. This gives a simple explanation to the question of what is the optimal number of wells, a question that is usually answered in vague terms, based on experience of analogous fields.

If one ignores abandonment (which is reasonable when one is trying to gain an understanding rather than calculate a specific value), NPV can be expressed as

$$NPV = R.L.\left(1 - \frac{1}{\frac{N.q}{R.d} + 1}\right) - N.\left(C + \frac{E}{d}\right) - D$$

The first R.L term (reserves x net oil price) corresponds to the value of the oil in the ground, i.e. the value of the field if the oil could be taken all at once at no cost.

The $\frac{R.L}{\frac{N.q}{R.d} + 1}$ term represents the value lost because of the discounting effect of the time taken to get the oil out of the ground. Clearly, if the number of wells is very large, this term becomes very small – the field is produced in a short space of time. On the other hand, if the number of wells is very small, then this term becomes almost as big as R.L, the total value of the oil – the field is produced very slowly, so the NPV of the production is very small.

The D term represents the value lost because of net fixed capex (roads, pipelines etc). The N.(C+E/d) term represents the value lost because of capex per well and opex. Clearly, this is linear with the number of wells.

Comparing a set of different scenarios, each with a different number of wells, it can be seen that the value of the oil in the ground and the net fixed capex do not vary, but the other two terms do. As the number of wells increases, the bulk of the oil is produced sooner, so less value is lost because of time effects, but opex and variable capex costs are increasing. The optimal number of wells is reached when the sum

(value lost in time effects) + (value lost because of opex and variable capex expenditure)

is at a minimum. This is illustrated in the plot below, for the case of a field with the following parameters:-

Technically recoverable reserves = 100 million barrels

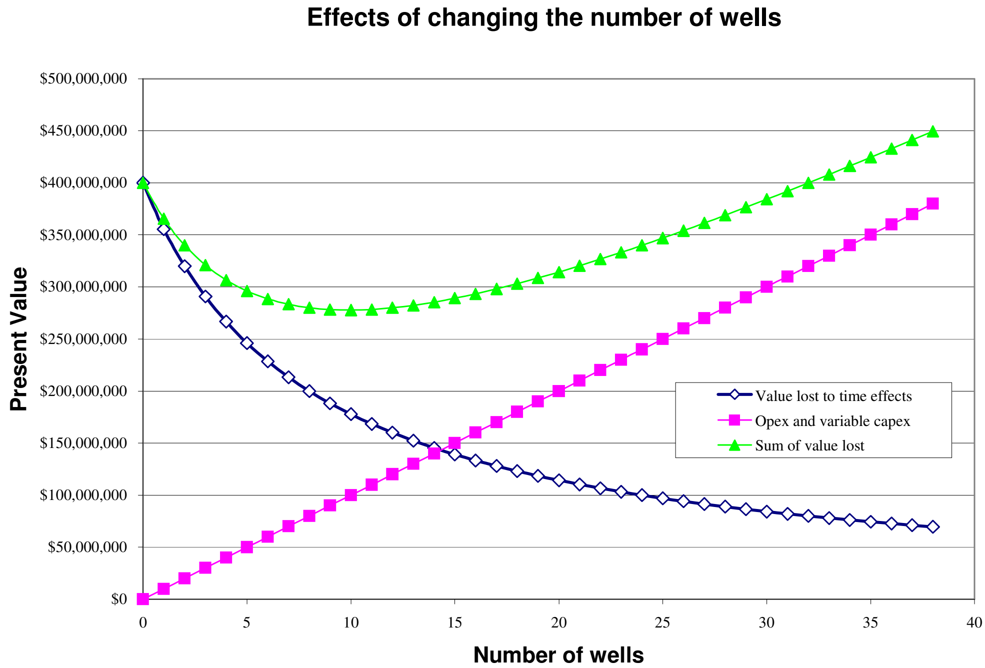
Average initial well rate = 2,800 bbl/day

Net capex per well = $5,000,000

Net opex per well = $400,000

Net oil price = $4.00

Discount rate = 8%



It can be seen that the optimal number of wells is about 10.

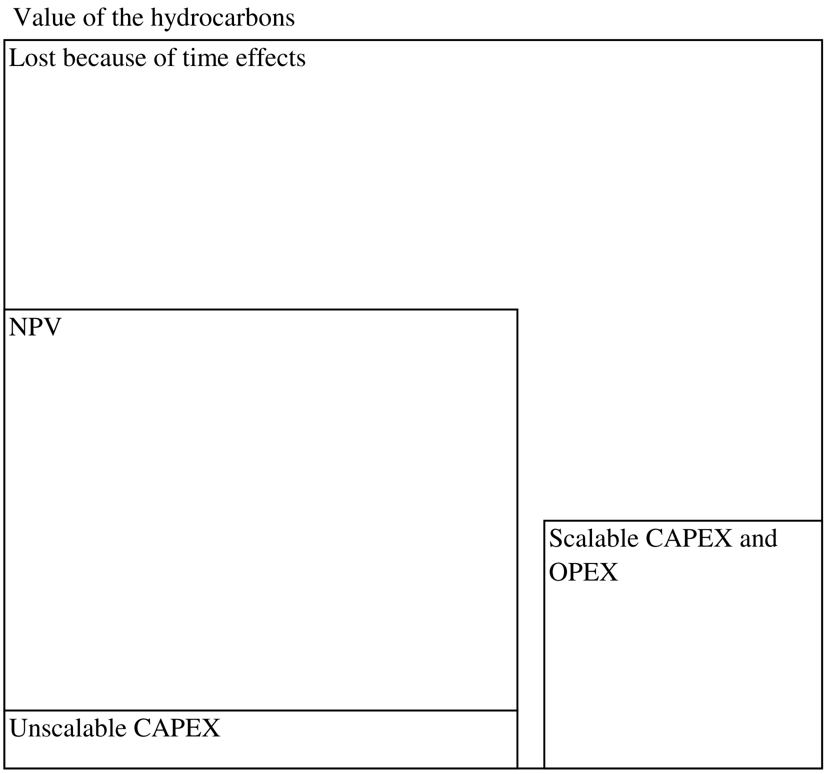
Visualising the NPV formula

To gain a feel for the significence of the NPV formula (for the optimal number of wells), it may be helpful to consider the formula as follows. Imagine a square whose area represents the total value (to the oil company) of the field, if all the oil could be produced immediately at zero cost i.e. R x L. Let the length of the sides of the square correspond to √L. If we ignore abandonment effects for the moment, then the scalable capex and the opex reduce the value to

$$R.\left(\sqrt{L}-\sqrt{\frac{C.d+E}{q}}\right)^2$$

This is the area of the square that can be fitted in the R.L square if there has been already inserted a square of length √[(C.d + E)/q].

Then the value has to be further reduced to pay for the fixed capex, D, represented by a suitable rectangle. The final step in building up the picture is to allow abandonment effects to introduce a gap between the two smaller squares.



Such a geometric representation is scarely any more intuitive than the formula itself, but it is put forward in the hope that it may a starting point for better visualisations or interpretations of the formula.

Using the formulas to derive the optimal injector:producer ratio

Another advantage of formulas is that they can be in further calculations. For example, in a water-flood, given constraints on well bottom-hole pressures, it is straight-forward, through a material balance argument, to express the expected average well rate (averaged over all wells, both producers and injectors) as a function of the injector:producer ratio, independently of the number of wells. Feeding this expression into the NPV formula gives an expression of NPV as a function of injector:producer ratio. This expression can be diffentiated w.r.t. the injector:producer ratio, to give a formula for the optimal injector producer ratio as follows (proof not given in this article)

Optimal injector:producer ratio = $\sqrt{\frac{PI.Bo.(Cp.d + Ep)}{II.Bw.(Ci.d + Ei)}}$

where

PI = productivity index of average production well

II = injectivity index of average injection well

Bo = oil formation volume factor

Bw = water formation volume factor

Cp = net capex cost per producer

Ci = net capex cost per injector

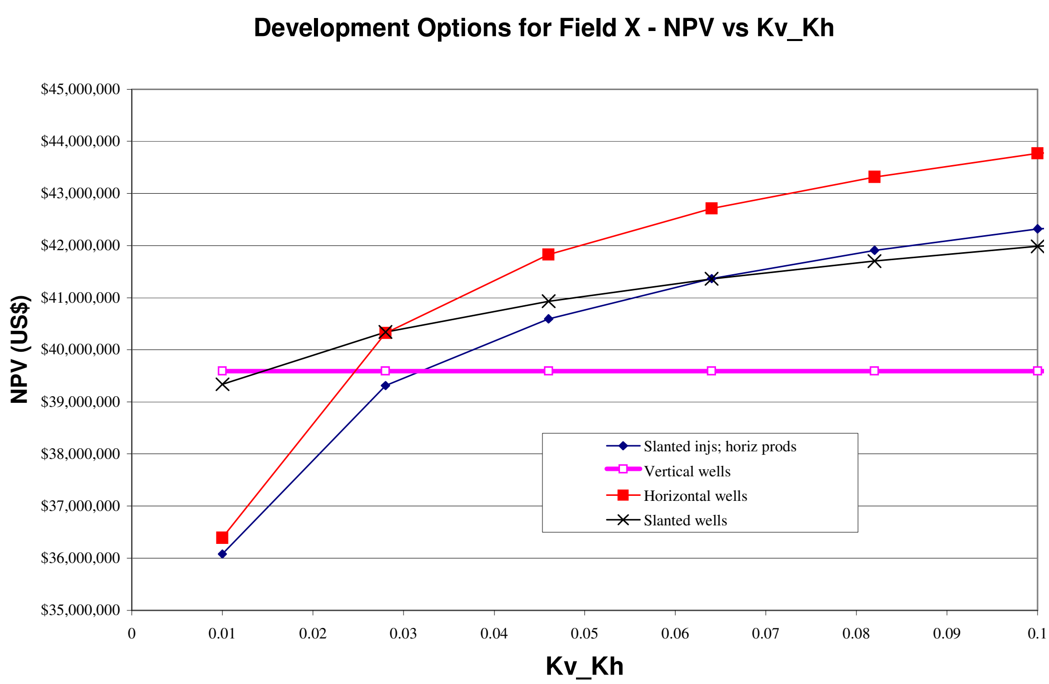
Ep = net opex per year for each producer

Ei = net opex per year for each injector

d = discount rate

Uncertainties in key parameters

By reducing the complexity of the economics calculations, the formulas make it possible to examine a wider range of uncertainties in key parameters. For example, it is easy to generate plots such as the one below, to compare the overall economic performance of horizontal and vertical wells, when there is a big uncertainty in the Kv:Kh ratio.



It can be seen that, for such a field, horizontal wells would be preferably to vertical wells providing the Kv/Kh ratio is greater than 0.02. Such analysis can help identify the key uncertainties to be addressed during field appraisal.

Extending the method to maximise rate of return

A simple modification to the Capex figures makes it possible to use the formulas when the company objective is to maximise overall return on capital employed.

The formulas as they stand show the number of wells required to maximise NPV. In the real world, oil companies are faced with the problem of constraints. In particular, the companies usually have only a fixed amount of capital available, or at least limits on the amount of capital employed (gearing limits etc). The problem then is to maximise NPV, subject to the limits on the amount of capital employed. Assuming that there are sufficient number of projects available and no other constraints intrude (such as limits on management capacity), the method to do this is to rank all the projects in order of decreasing NPV:Capex ratio, and then choose the projects in this order until one has reached the limit on the amount of capital employed.

For the projects chosen, let H (the "hurdle" rate) be the lower limit to the NPV:Capex ratio i.e. the NPV:Capex ratio for the last project chosen.To determine, for a new project, the optimal number of wells, it is useful to consider the project as consisting of

(Minimalist development option) + Series of increments to the mininalist development option.

(Note that the split is purely conceptual; all the increments start at time = 0).

All the increments can be analysed as if they were separate projects. They pass the screening criteria if their NPV:Capex ratio is greater than H

i.e. $\delta NPV / \delta Capex \geq H$

So, while the NPV/Capex ratio for additional wells is greater than H, the wells are worth adding to the development scheme. The optimal total number of wells is reached when the limit is reached, so the criteria for determining the total number of wells is

$\delta NPV / \delta Capex = H$

This can be converted into a more workable criteria as follows

$\delta NPV / \delta Capex = H$

$\Leftrightarrow \delta(NPV-H.Capex) / \delta Capex = 0$

$\Leftrightarrow [\delta(NPV-H.Capex) / \delta N] \times [\delta N / \delta Capex] = 0$ where N is the number of wells

$\Leftrightarrow \delta(NPV-H.Capex) / \delta N = 0$ since $\delta N / \delta Capex \neq 0$ (it never costs an infinite amount of money to drill a new well).

Hence, the new problem (find the number of wells that maximises corporate NPV subject to a limit on capital employed) can be converted into the old (find the number of wells that maximises NPV) by using an artifical NPV, defined to be

$NPV' = NPV - H.Capex = NPV - H.(C.N + D)$

and artificial C′ and D′ defined to be

C′ = (1 + H) x C

D′ = (1 + H) x D

Then

$$\text{Optimal number of wells, } N = \frac{R.d}{q} . \left(\sqrt{L - \frac{\alpha.E}{q}} - \sqrt{\frac{C'.d + E - \alpha.E}{q}} \right) \Bigg/ \sqrt{\frac{C'.d + E - \alpha.E}{q}}$$

$$\text{NPV with this number of wells} = NPV' + H.(C.N + D) = R.\left(\sqrt{L - \frac{\alpha.E}{q}} - \sqrt{\frac{C'.d + E - \alpha.E}{q}} \right)^2 + H.C.N - D$$

(the –H.D and +H.D terms cancel each other out).

Dealing with a development spread over several years

If a field is very large, so that the development is spread over several years, with different areas coming on stream in different years, then the situation can be modelled as follows.

Let NPV(N) be the NPV that would have been achieved if the development had been had carried out in the space of one year.

Let M be the number of years that will be required. (Assume M to be independent of N, the number of wells).

The new problem is to find N to maximise

$$NPV' = \sum_{i=1}^{M} \frac{NPV(N)}{M} (1 + d)^{-(i-1)} = \frac{(1+d) - (1+d)^{-(M-1)}}{M.d} .NPV(N)$$

Clearly, the value of N that maximises NPV(N) also maximises NPV′ and the above equation gives NPV′. So the problem is solved.

Appendix – Mathematical proofs of the formulas

Assumptions and Definitions

Let us make the following assumptions to describe an oil-field development

1) Let N be the number of wells drilled. All the wells start up at time t=0.
2) If the field were run for an infinite time, the total production would be R (the technically recoverable reserves), independent of the number of wells.
3) The initial production rate per well is q, independent of the number of wells i.e. it is not affected by well spacing.

4) The field oil production rate follows exponential decline i.e.
   Field oil production rate = initial rate x e^{-at} = $N.q.e^{-at}$
5) The net oil price is a constant, L, after all taxes and deductions.
6) The net capital costs can be expressed as D + C.N; all capital expenditure happens at time t=0.
7) Net opex can be expressed as E.N per unit time. (The year is probably the most appropriate unit of time, but any unit can be used, providing it is the same for both E and q).

Theorem 1 – Expressing NPV as a function of well numbers

The NPV of a field run until abandonment can be expressed as

$$\mathrm{NPV} = \frac{\mathrm{R.L}}{1+\left(\mathrm{R.\ln(1+d)}\big/\mathrm{q.N}\right)}.\left(1-\alpha.e^{-\mathrm{q.N.Tab/R}}\right) - \frac{\mathrm{N.E}}{\ln(1+d)}.(1-\alpha)-(\mathrm{C.N}+\mathrm{D})$$

Proof

NPV can be broken down into the component parts of the cash-flow

NPV = NPV(revenue stream) + NPV(Opex) + NPV(Capex)

where the NPV(Opex) and NPV(Capex) are, of course, negative.

Start by calculating the NPV of the revenue stream:-

By the assumption that there is exponential decline

Oil production rate = $N.q.e^{-at}$ = $N.q.e^{-(q.N/R)t}$

[Since, by the definition of technical reserves

$$\mathrm{R} = \int_0^\infty \mathrm{N.q.e}^{-a.t}\,dt = \left(-\frac{\mathrm{N.q}}{a}.e^{-at}\right]_0^\infty = \mathrm{N.q}/a$$

then a = N.q/R]

Oil revenue per unit time = (production rate) x (net oil price) = $N.L.q.e^{-q.N.t/R}$

By the definition of NPV

$$\text{NPV of revenue stream} = \int_0^{Tab} \frac{\text{Revenue per unit time}}{(1+d)^t} dt = \int_0^{Tab} N.L.q.e^{-(q.N/R+\ln(1+d)).t} dt$$

$$(\text{Since } (1+d)^t = e^{\ln((1+d)^{-t})} = e^{-t.\ln(1+d)})$$

$$= \left. \frac{N.L.q.e^{-(q.N/R+\ln(1+d))t}}{-(\frac{q.N}{R}+\ln(1+d))} \right|_0^{Tab} = \frac{N.L.q}{\left(\frac{q.N}{R}+\ln(1+d)\right)}.\left(1-e^{-(q.N/R+\ln(1+d)).Tab}\right)$$

$$= \frac{R.L}{\left(1+\frac{R.\ln(1+d)}{q.N}\right)}.\left(1-e^{-\ln(1+d).Tab}.e^{-q.N.Tab/R}\right) = \frac{R.L}{\left(1+\frac{R.\ln(1+d)}{q.N}\right)}.\left(1-(1+d)^{-Tab}.e^{-q.N.Tab/R}\right)$$

$$= \frac{R.L}{\left(1+\frac{R.\ln(1+d)}{q.N}\right)}.\left(1-\alpha.e^{-q.N.Tab/R}\right)$$

Looking at the Opex cash-flow

Opex per unit time = -N.E

$$\text{NPV of opex} = \int_0^{Tab} \frac{-N.E}{(1+d)^t} dt = -\int_0^{Tab} N.E.e^{-\ln(1+d).t} dt = \left. \frac{N.E.e^{-\ln(1+d).t}}{\ln(1+d)} \right|_0^{Tab} = -\frac{N.E}{\ln(1+d)}.\left(1-e^{-\ln(1+d).Tab}\right)$$

$$= -\frac{N.E}{\ln(1+d)}.\left(1-(1-d)^{-Tab}\right) = -\frac{N.E}{\ln(1+d)}.(1-\alpha)$$

Looking at Capex, since all the capital expenditure is assumed to occur at time t=0,

NPV of Capex = - (C.N + D)

By adding NPV(Revenue), NPV(Opex) and NPV(Capex), one obtains the desired formula.

Q.E.D

Theorem 2 – Number of wells giving the highest NPV

Part I

The number of wells giving the highest NPV, N_{opt}, can be calculated (iteratively) from the expressions

$$Nopt = \frac{R}{q}.\ln(1+d).\left[\sqrt{\frac{L.q-\alpha.E}{C.\ln(1+d)+E-\alpha.E}}x\left(\sqrt{\varepsilon^2+1}-\varepsilon\right)-1\right]$$

$$\text{where } \varepsilon = \frac{\alpha.E.\ln\left(\frac{L.q}{E}\right)}{\sqrt{4(L.q-\alpha.E).(C.\ln(1+d)+E-\alpha.E)}}$$

$$\text{and } \alpha = (1+d)^{-Tab}$$

$$\text{and } Tab = \frac{R}{q.N}.\ln\left(\frac{L.q}{E}\right)$$

Part II

If

a) $L.q \geq 2.[C.\ln(1+d) + E]$ (i.e. the well is reasonably profitable at first) and
b) $0.2 \geq \alpha$ (equates, for d = 8%, to a field life ≥ 20 years)

then the approximation of setting $\varepsilon = 0$ gives approximate value for the number of wells, N_{approx} within the following bounds

$$1.31\ N_{opt} \geq N_{approx} \geq N_{opt}$$

Proof of Part I

Consider NPV as a function of well numbers and abandonment time. For a normally behaved function

$$\text{NPV is a maximum} \Leftrightarrow \frac{\partial NPV}{\partial N} = \frac{\partial NPV}{\partial Tab} = 0$$

$$\text{Tab such that } \frac{\partial NPV}{\partial Tab} = 0$$

can be obtained by taking the partial derivatives or, more simply, by seeing that this occurs when revenue has dropped until it equals operating expenditure

i.e. $N.L.q.e^{-q.N.Tab/R} = N.E$

so $-q.N.T_{ab}/R = \ln(E/(L.q))$

so $Tab = R/(q.N)\ .\ \ln(L.q/E)$

(N.B Also, for Tab optimal, $e^{-q.N.Tab/R} = E/(L.q)$)

For well numbers, using Theorem 1, and writing out the expression in full (not using α, for example)

$$\frac{\partial NPV}{\partial N} = \frac{\partial}{\partial N}\left(N.\left[\frac{Lq}{\frac{qN}{R} + \ln(1+d)}.\left(1-(1+d)^{-Tab}.e^{-(qNTab/R)}\right) - \frac{E(1-(1+d)^{-Tab})}{\ln(1+d)} - C \right] - D \right)$$

$$= \frac{Lq}{\frac{qN}{R} + \ln(1+d)}.\left(1-(1+d)^{-Tab}.e^{-(qNTab/R)}\right) - \frac{E(1-(1+d)^{-Tab})}{\ln(1+d)} - C$$

$$- \frac{N.Lq}{\left(\frac{qN}{R} + \ln(1+d)\right)^2}.\left(1-(1+d)^{-Tab}.e^{-(qNTab/R)}\right).\frac{q}{R}$$

$$+ \frac{N.Lq}{\frac{qN}{R} + \ln(1+d)}.\left(1-(1+d)^{-Tab}.e^{-(qNTab/R)}\right).\left(-\frac{q.Tab}{R}\right)$$

Setting this partial derivative equal to zero, multiplying both sides of the resultant equation by $(qN/R + \ln(1+d))^2$, then using α as shorthand for $(1+d)^{-Tab}$, and the relationships that apply when T_{ab} is optimal, $e^{-q.N.Tab/R} = E/(L.q)$ etc, we get the following equation. N.B. This equation only applies when T_{ab} is optimal.

$$\left(\frac{q.N}{R} + \ln(1+d)\right).L.q.\left(1-\alpha.\frac{E}{L.q}\right) - \left(\frac{q.N}{R} + \ln(1+d)\right)^2.\left(\frac{E.(1-\alpha)}{\ln(1+d)} + C\right) - \frac{qN}{R}.L.q.\left(1-\alpha\frac{E}{L.q}\right)$$

$$+ \ln\left(\frac{L.q}{E}\right).L.q.\left(\frac{q.N}{R} + \ln(1+d)\right).\alpha.\frac{E}{L.q} = 0$$

Cancelling terms and multiplying both sides by –1 gives

$$\left(C + \frac{E.(1-\alpha)}{\ln(1+d)}\right).\left(\frac{q.N}{R} + \ln(1+d)\right)^2 + \alpha.E.\ln\left(\frac{Lq}{E}\right)\left(\frac{q.N}{R} + \ln(1+d)\right) - \ln(1+d).(L.q - \alpha.E) = 0$$

This can be considered to a quadratic equation, with the “x” term being (q.N/R + ln(1+d)) and the other terms being as follows:-

$a = C + E.(1-\alpha)/(\ln(1+d)$

$b = \alpha.E.\ln(L.q/E)$

$c = -\ln(1+d).(L.q - \alpha.E)$

We will proceed here in Part I of this proof to solve the quadratic equation. In Part II, we will show that the “b” term has little effect, and can be ignored. (It is interesting to consider how the “b” term arose. Consider the NPV of the field. If the number of wells increases, then abandonment is brought forward, so the term representing the present value of the oil lost at abandonment is increased. This decreases the overall NPV, but as can be imagined, such effects are small. This will be proved later, in Part II).

So, ignoring the negative solution, the solution of the quadratic equation is:

$$x = \frac{\sqrt{b^2 - 4a.c} - b}{2a}$$

Re-arranging this gives

$$x = \sqrt{\frac{-c}{a}}.\left[\sqrt{1 + \left(\frac{b}{\sqrt{-4a.c}}\right)^2} - \frac{b}{\sqrt{-4a.c}}\right]$$

$$\text{Defining } \varepsilon \text{ by } \varepsilon = \frac{b}{\sqrt{-4a.c}} = \frac{\alpha.E.\ln\left(\frac{L.q}{E}\right)}{\sqrt{4.(C.\ln(1+d) + E - \alpha.E).(L.q - \alpha.E)}}$$

and expanding x, c and a gives

$$\frac{q.N}{R} + \ln(1+d) = \sqrt{\frac{\ln(1+d).(L.q - \alpha.E)}{C + \frac{E.(1-\alpha)}{\ln(1+d)}}}.\left(\sqrt{\varepsilon^2 + 1} - \varepsilon\right)$$

$$= \ln(1+d)\sqrt{\frac{(L.q - \alpha.E)}{C.\ln(1+d) + E - \alpha.E}}.\left(\sqrt{\varepsilon^2 + 1} - \varepsilon\right)$$

Re-arranging this equation gives

$$N = \frac{R}{q}.\ln(1+d).\left[\sqrt{\frac{(L.q - \alpha.E)}{C.\ln(1+d) + E - \alpha.E}}.\left(\sqrt{\varepsilon^2 + 1} - \varepsilon\right) - 1\right]$$

Proof of Theorem 2, Part II

As a first step, it is useful to note that, for $\varepsilon \geq 0$ (which is the case, providing $L.q \geq E$ – i.e. Year 1 net revenue for a well is greater than the opex for the well)

$1 \geq (\sqrt{(\varepsilon 2+1)} - \varepsilon) \geq 1-\varepsilon$ hence, Napprox ≥ Nopt

$$[\text{ Proof - } \left(\sqrt{\varepsilon^2 + 1} - \varepsilon\right)^2 = \varepsilon^2 + 1 - 2\varepsilon\sqrt{\varepsilon^2 + 1} + \varepsilon^2 = 1 + 2\varepsilon.\left(\varepsilon - \sqrt{\varepsilon^2 + 1}\right) \leq 1$$

$$\text{since } \sqrt{\varepsilon^2 + 1} \geq \varepsilon\ ]$$

Examining ε^2 and dividing both the denominator and quotient by $(L.q)^2$ gives

$$\varepsilon^2 = \frac{(\ln(L.q/E))^2 \alpha^2 (E/L.q)^2}{4\left[\frac{C.\ln(1+d)+(1-\alpha).E}{L.q}\right]\left(1-\frac{\alpha E}{L.q}\right)}$$

$$\text{Since } \frac{E}{L.q} \le \frac{C.\ln(1+d)+E}{L.q} \le 0.5 \text{ (by assumption 1) and } \alpha \le 0.2 \text{ (by assumption 2)}$$

$$\left(1-\frac{\alpha E}{L.q}\right) \ge 1-0.2x0.5 = 0.9$$

Also

$$\frac{C.\ln(1+d)+(1-\alpha).E}{L.q} \ge (1-\alpha).\frac{E}{L.q} \ge 0.8\frac{E}{L.q}$$

Hence,

$$\varepsilon^2 \le \frac{(\ln(L.q/E))^2 0.2^2 (E/L.q)^2}{4x0.8x(E/L.q)x0.9} = 0.014(\ln(L.q/E))^2.\frac{E}{L.q}$$

It can be easily shown that for 1 ≥ E/(L.q) ≥ 0, the maximum value of $(\ln(L.q/E))^2.E/(L.q)$ is achieved when E/(L.q) = $1/(e^2)$ = 0.1353, which gives $(\ln(L.q/E))^2.E/(L.q)$ = 0.541.

[Proof - Differentiate $x.(\ln(x))^2$ and set to zero].

Hence ε ≤ 0.014 x 0.541 = 0.00757

ε ≤ 0.087

$(\sqrt{(1+\varepsilon^2)} - \varepsilon) \ge (1-\varepsilon) \ge (1-0.087) = 0.93$

Before moving on to look at a lower limit for N_{opt}/N_{approx}, it is useful to establish a couple of small lemmas.

Lemma A

For u, v, w such that u ≥ v > 0 and v > w ≥ 0,

$$\frac{u-w}{v-w} \ge \frac{u}{v}$$

$$\text{Proof } \frac{u-w}{v-w} - \frac{u}{v} = \frac{uv - vw - uv + uw}{v(v-w)} = \frac{(u-v).w}{v.(v-w)} \ge 0$$

Lemma B

For w constant and greater than zero, the function f (y) = [(1-w).y −1] / (y-1) is strictly increasing (i.e. $y_1 < y_2 \Rightarrow f(y_1) < f(y_2)$).

Proof Expressing f (y) as 1 – wy / (y-1) gives

$$\frac{df}{dy} = \frac{-w}{y-1} + \frac{w.y}{(y-1)} = \frac{w}{(y-1)} > 0$$

Moving back to the main proof, let us establish a lower bound on N_{opt}/N_{approx}

$$\frac{Nopt}{Napprox} = \frac{\sqrt{\frac{Lq-\alpha.E}{C.\ln(1+d)+E-\alpha.E}}.\left(\sqrt{1+\varepsilon^2}-\varepsilon\right)-1}{\sqrt{\frac{Lq-\alpha.E}{C.\ln(1+d)+E-\alpha.E}}-1} \geq \frac{0.93\sqrt{\frac{Lq-\alpha.E}{C.\ln(1+d)+E-\alpha.E}}.-1}{\sqrt{\frac{Lq-\alpha.E}{C.\ln(1+d)+E-\alpha.E}}-1}$$

By Lemma A and assumption (i)

$$\sqrt{\frac{Lq-\alpha.E}{C.\ln(1+d)+E-\alpha.E}} \geq \sqrt{\frac{Lq}{C.\ln(1+d)+E}} \geq \sqrt{2}$$

By Lemma B

$$\frac{0.93\sqrt{\frac{Lq-\alpha.E}{C.\ln(1+d)+E-\alpha.E}}.-1}{\sqrt{\frac{Lq-\alpha.E}{C.\ln(1+d)+E-\alpha.E}}-1} \geq \frac{0.93\sqrt{2}-1}{\sqrt{2}-1} = 0.76$$

Hence $N_{opt}/N_{approx} \geq 0.76$

or equivalently $N_{approx} \leq 1.31 N_{opt}$

Combining this result with the result established at the beginning of the Part II of the proof gives

$$1.31\ N_{opt} \geq N_{approx} \geq N_{opt}$$

Q.E.D.

Note – In order to express the expression for the approximate optimal number of wells in the form first quoted

$$\text{Number of wells that maximises NPV} = \frac{R.d}{q}.\left(\sqrt{L-\frac{\alpha.E}{q}} - \sqrt{\frac{C.d+E-\alpha.E}{q}}\right) \bigg/ \sqrt{\frac{C.d+E-\alpha.E}{q}}$$

it suffices to rearrange the expression slightly and to note that for normal values of d (the discount rate), ln(1+d) ≈ d (e.g. ln(1+0.08) = 0.077)

Theorem 3 – NPV for a development with the approximately optimal number of wells

The NPV of a development with the approximately optimal number of wells, as defined in Theorem 2, is

$$NPV = R.\left(\sqrt{L - \frac{\alpha.E}{q}} - \sqrt{\frac{C.\ln(1+d) + (1-\alpha).E}{q}}\right)^2 - D$$

where α = (1+d)-Tab

and Tab = R / (N.q) . ln(L.q/E)

Proof

Combining the results from Theorems 1 and 2

$$NPV = \frac{R.\ln(1+d)}{q}\left(\sqrt{\frac{L.q - \alpha.E}{C.\ln(1+d)+(1-\alpha).E}} - 1\right) x$$

$$\left[\frac{L.q}{\frac{q.R.\ln(1+d)}{R.q}.\left(\sqrt{\frac{L.q-\alpha.E}{C.\ln(1+d)+(1-\alpha).E}} - 1\right) + \ln(1+d)}.\left(1 - \frac{\alpha.E}{L.q}\right) - \frac{E.(1-\alpha)}{\ln(1+d)} - C\right]$$

$$-D$$

$$= R.\left(\frac{\sqrt{a}}{\sqrt{b}} - 1\right).\left[\frac{L}{\left(\sqrt{a}/\sqrt{b}\right)}.\left(1 - \frac{\alpha.E}{L.q}\right) - \frac{E.(1-\alpha)}{q} - \frac{C.\ln(1+d)}{q}\right] - D$$

(where a = L.q - α.E

and b = C.ln(1+d) +(1-α).E)

$$= R.\left(\frac{\sqrt{a}}{\sqrt{b}} - 1\right).\left[\frac{a.\sqrt{b}}{q.\sqrt{a}} - \frac{b}{q}\right] - D = R.\left(\sqrt{\frac{a}{q}} - \sqrt{\frac{b}{q}}\right).\left[\sqrt{\frac{a}{q}} - \sqrt{\frac{b}{q}}\right] - D$$

$$= R.\left(\sqrt{L - \frac{\alpha.E}{q}} - \sqrt{\frac{C.\ln(1+d)+(1-\alpha).E}{q}}\right)^2 - D$$

Q.E.D.